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14. ABSTRACT
According to the Kelvin's formula paradox, a polarized body will be accelerated by its own electrostatic or magnetostatic field. This paradoxical statement obviously contradicts common sense and everyday experience, and should be eliminated from any rational theory of electromagnetism. We suggested a general approach allowing to get rid of this paradox. However, the approach leads to quite complex formulae. Needless to say, a simpler resolution of the paradox, if possible, would be highly desirable. A potentially simpler resolution of the paradox was recently suggested by our colleagues. According to their suggestion, proper account of the mechanical stresses in the body compensates the resultant polarization force. In this report, we demonstrate, however, that the suggested mechanism of compensation is impossible. Basically, we reinforce the concept that no redistribution of stresses is able to compensate the volume force if the resultant force does not vanish. In other words, with a nonvanishing resulting force, the body is not able to be at rest no matter what its constitutive equation will be.

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1. Preamble

Electromagnetic forces provide us with various phenomena and mechanisms of key importance for applications, including those which are of interest for Army science. Because of their importance, the key notions of electromagnetism should be elucidated with maximum clarity. Among those notions of key importance are the forces, energy, and entropy of metallic and polarizable substances. Yet, these key notions remain unclear and, as such, repeatedly trigger hot debates.[1]

Analysis of the key concepts and notions of any research discipline is part of the validation and verification (V&V) procedures. In particular, the V&V of electromagnetic models, techniques, and computer codes continues to gain more attention in current engineering and fundamental research. Analysis of the electromagnetic forces is an essential part of the fundamental research. In recent publications,[2,3] we suggested a novel, mostly thermodynamic, analysis of the electromagnetic forces, acting in polarizable materials. When fulfilling those V&V studies of electromagnetic forces, we faced unexpectedly a paradoxical outcome of the famous Kelvin formula of the ponderomotive forces—both electrostatic and magnetostatic.

We remind the readers that according to Kelvin's formula, the total force $\vec{F}_{res}$, acting on the electrically polarized body, is given by the formula

$$\vec{F}_{res} = \int_{\Omega} d\Omega \vec{P} \cdot \nabla \vec{E}, \tag{1}$$

where $\vec{E}$ is the electric field, $\vec{P}$ is the polarization density, and the integration in (1) is taken over the whole polarized body. The elegant formula (1) is intuitively transparent and simple. Not surprisingly, it has become the "working horse" of multiple engineering applications and academic studies, yet it carries an essential conceptual defect—namely, the resultant self-force, acting on a permanent magnet from the magnetic field, generated by the magnet itself, does not vanish (see the proof in Grinfeld and Grinfeld[2]). Simply speaking, the magnet will be accelerated by its own magnetic field; this conclusion is in striking disagreement with our everyday experience.

In a recent publication,[3] we suggested the approach to eliminate the abovementioned self-action paradox. However, the suggested variational approach leads to a much more complicated and cumbersome analysis and formula. One simple treatment has been suggested by our colleagues. Actually, they strongly insist that our paradoxical conclusion is based on the fact that we completely

ignored the role of mechanical stresses. They claim that proper distribution of stresses is able to compensate the resultant nonzero self-acting polarization force.

In fact, it is not true that we ignored or overlooked the role of mechanical stresses. We strongly believed that their discussion has no relevance to the resultant force and its influence. We believe that no internal mechanical stresses are able to compensate the self-acceleration brought by the influence of nonvanishing resulting force. In fact, no internal stresses, regardless of the constitutive model of the substance, can resolve the Kelvin's formula paradox.

Our colleagues are definitely right in one respect: the internal stresses, indeed, may have crucial consequences for the behavior of solids. There is, though, one influence that the internal forces are not able to produce. Namely, *they are not able to generate a nonvanishing resultant force—their resultant force always vanishes due to Newton's third law*. In the Kelvin's formula paradox, namely, the resultant polarization force is the one and only object of the discussion.

In the following section, we remind the readers interested in the Kelvin's formula paradox of this fundamental "disability" of internal mechanical stresses. It is highly desirable to find another simple justification and treatment of this paradox.

2. Stresses and Resultant Forces

Consider any material body that is under the action of volumetric force $P^i(x)$. Under the action of those forces the stresses σ^{ij} will be generated inside the body. If the acceleration vanishes the following equilibrium equation must be satisfied at each point of the body

$$\frac{\partial \sigma^{ij}}{\partial x^j} = P^i \quad . \tag{2}$$

We assume, like in the case of free magnet, that there are no surface forces acting on the body through its boundary s. Mathematically, this condition reads

$$\sigma^{ij} n_j = 0 \quad , \tag{3}$$

where n_j are the components of unit normal of the body's surface, s.

Let us integrate the relationship (2) over the whole volume ω occupied by the body. We then get

$$\int_\omega d\omega \frac{\partial \sigma^{ij}}{\partial x^j} = R^i \quad , \tag{4}$$

where

$$R^i = \int_\omega d\omega P^i \tag{5}$$

is the so-called resultant force acting on the body.

Let us look closer at the left-hand side of Eq. 4. The so-called divergence, or the Gauss-Ostrogradsky formula, allows one to replace the integral over the volume ω with the integral over the boundary surface s:

$$\int_\omega d\omega \frac{\partial \sigma^{ij}}{\partial x^j} = \int_s ds \sigma^{ij} n_j \ . \tag{6}$$

But according to the boundary condition (3), the integral on the right-hand side of (6) vanishes. Therefore, the integral on the left-hand side of the relationship (4) vanishes as well, and we arrive at the desired relationship for the resultant force

$$R^i = 0 \ . \tag{7}$$

Let us now discuss the implication of the relationship (7). First, we see that if the resultant force R^i does not vanish, $R^i \neq 0$, our starting assumption (that the internal stresses are able to compensate the material forces and keep the body at rest) is wrong. This conclusion does not depend on the physical nature of volumetric forces P^i—they can be magnetostatic, electrostatic, or have any other nature. Equally, our conclusion does not depend on the constitutive model (i.e., upon the dependence of stresses σ^{ij} on the deformations or their history [the so called, rheological equations]).

3. Conclusions

The Kelvin force paradox, formulated in Grinfeld and Grinfeld[2], is equally applicable to rigid (i.e., nondeformable) solids as well as to elastic, plastic, liquid, or gaseous substances. In our formulation of the paradox in Grinfeld and Grinfeld[2], we did not discuss anything relating to stresses. Such a discussion would be just a misleading distraction. Indeed, it did not make sense to discuss internal mechanical stresses when discussing only resultant forces: the contribution of mechanical stresses of any nature in the resultant force is always zero. In the paper by Grinfeld and Grinfeld[3], we suggested the formulas for the forces acting in polarizable substances. The formulas of Grinfeld and Grinfeld[3] are free of the Kelvin's formula paradox. But they are much more complicated. Therefore, in no way do we recommend eliminating Kelvin's formula right away from the current studies. At this stage, we only recommend to remember the conceptual problems associated with the Kelvin's formula.

4. References

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